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New Stochastic Programming Approximations to Network Capacity Control Problem with Buy-ups

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Abstract

It is well known that the network capacity control (NCC) problem can be formulated by a dynamic programming model (see, for example, Talluri and van Ryzin [30]). However, this formulation is unsolvable in practice due to its size and complexity. As a result, various approximation methods are proposed. Decomposition and deterministic linear programming approximations are formulated and have successfully been utilized in practice. Most recently, several stochastic programming approaches, which incorporate demand uncertainty into account, are published. This article adds to the recent research on stochastic programming methodologies by considering the customer's buy-up behavior. It provides three new formulations based on different sets of assumptions and simulates a demand arrival process to evaluate the performance of each approach as they compare with the deterministic linear programming approximation.

Keywords: Network capacity control, network resource allocation, stochastic programming, customer choice models

1 Introduction

Revenue management (RM) involves use of sophisticated procedures to balance supply and demand to maximize revenue and profit growth. RM has generated billions of additional dollars for many industries including airlines, hotels, car rental firms, cruise lines, utility firms, railroads, retailers, tour operators, manufacturers, and media companies (e.g., Cross [16], Smith et al. [27]). Two recent books, Talluri and van Ryzin [30] and Phillips [25], provide comprehensive treatments of revenue management problems.

Broadly speaking, RM utilizes two types of procedures: *pricing* and *inventory control*[1]. The main goal of pricing procedures is to isolate and measure the effects of price and optimize it across different micro-segments[2]. On the other hand, inventory control focuses on demand that should be accepted or rejected at a given price and micro-segment. In general, inventory control methods are effective when capacity has a primary influence in prices; otherwise, pricing methods are more relevant. This article considers inventory control problem.

The inventory control can be viewed as a special class of the problem of matching demand to supply. In this view, supply is known, difficult to adjust, and limited[3]. Demand, on the other hand, is highly uncertain and stochastic. This is because of a number of factors, such as consumption and purchase times, price, quality and availability of supply, and competing products. This article assumes that demand is stochastic.

Inventory control methods assume single or multiple resources. Single resource models consider each resource independent of each other. Multiple resource models consider all resources simultaneously, which is often referred as network capacity control (NCC). This article considers multiple resources.

Inventory control is accomplished by partitioned or nested allocation systems. In a partitioned allocation system, reservation requests for any fare product will be denied if the booking limit of the fare class is exceeded. In a nested seat allocation system, on the other hand, fare classes can be ordered in a strict hierarchy according to their revenue contributions, and a request for a fare product is always accepted as long as inventory is available for any lower fare products. This article develops three models; two models assume

[1] *Inventory control* is also referred as *allocation* and *demand rationing*

[2] Micro-segments are typically characterized by attributes of customers, products, orders, and time.

[3] In fact, supply is always limited, but in some circumstances, it does not directly impact pricing decisions for all practical purposes. For example, distributors do not consider availability a key issue for most products as they can always buy more if they can sell more.

partitioned allocations and the remaining one assumes nested allocations.

A key aspect of most available inventory control models is the assumption that demand is independent of inventory controls being applied by the seller. That is, the most work assumed that the customers who buy at full price are separate from the ones who buy at discount fares. However, the likelihood of customers buying a full fare increases when the discount fare is unavailable. The models proposed in this article consider dependence on the available inventory controls and allow buy-ups.

Current available optimization procedures capture important components of NCC problems, although no one considered a NCC problem with stochastic demand, partitioned or nested controls, and buy-ups simultaneously. Earlier studies that are most relevant to our research is Higle and Sen [20], Chen and Homem-de-Mello [13] and van Ryzin and Vulcano [33].

This article proposes three stochastic programming formulations with buy-ups. The first model assumes that the company is controls their inventory with partitioned booking limits, and customers can buy-up to any higher valued products available. The second model still assumes partitioned booking limits; however, customers leave the system if after they buy-up to the next higher fare product and still cannot purchase the product. The third model considers nested booking limits with the assumptions that customers can buy up to any higher valued product (like the first model). Simulation methodology is applied to evaluate each approach.

Although the airline specific terminology will used in our discussion, the proposed methodologies has a potential utilization in other service industries, such as lodging, car rental, printing and publishing, delivery service, food service, broadcasting, entertainment, health care, cruise line, trucking, and railroads industries.

The remainder of this article is organized as follows. Section 2 gives a brief overview of the related literature. Section 3 introduces the notation and the proposed stochastic programming models. Section 4 discusses the computational issues in solving these models. Section 5 provides simulation methodology and computation results. Section 6 gives the implementation issues. Finally, Section 7 contains a summary, conclusion of the work, and directions for further research.

2 Review of Related Literature

This section presents an overview of existing methodologies regarding NCC and customer choice models. It has been divided into five parts. The first part includes approximation methods using deterministic linear programming models. The second part discusses the decomposition methods. The third part discusses stochastic programming approximations. The fourth part presents the customer choice literature. The final part gives exact contribution of this article.

NCC problem can be formulated and solved as a dynamic programming model. Although this model is very useful to analyze the structural properties of the problem, it is unsolvable in practice due to its size and complexity. As a result, approximation methods are proposed. Deterministic linear programming (DLP) is the most well-known and commonly used technique in practice and originally formulated by D'Sylva [17] and Glover et al. [19]. These formulations produce partitioned booking limits, and the demand is approximated by its expected value. The dual variables corresponding to constraints are often utilized to implement bid-price controls. An alternative approach to approximate bid prices from DLP formulation is proposed by Bertsimas and Popescu [9].)

A major drawback of DLP approach is that it utilizes a single number, the expected demand, to characterize the demand. To circumvent this problem, a probabilistic nonlinear programming (PNLP) is proposed by Wollmer [36]. Even though probability structure explicitly formulated in PNLP, Williamson [35] shows that if optimized frequently, DLP will generally outperform PNLP. Talluri and van Ryzin [28] proposed a randomized linear programming model that generates random demand to solve each linear programming problem. The sample mean of the generated bid-prices is used as an approximation for the real bid-prices. In stochastic programming terminology, this is basically averaging over the so-called wait-and-see solutions. In theory, this approach can be further improved by using stochastic programming appropriately. For a discussion on comparison of wait-and-see solution and stochastic programming solution, we refer the reader to Birge and Louveaux [10].

Another approach for network problems is to decompose them into multiple single leg problems via methods like virtual nesting (see Smith et al. [27]). Then the problem is solved by using leg-based inventory control methods (e.g., Belobaba [4], [6]). Another class of methods use gradients generated via simulation. Bertsimas and de Boer [8] proposed a stochastic gradient procedure for discrete demand case and van Ryzin and Vulcano [32] gave a similar approach for continous demand. Recent books by Talluri and van Ryzin [30] and Phillips [25] provide an excellent treatment of NCC problem and further literature review on the subject.

Recently, there has been a considerable effort to formulate the network capacity conrol problem as a stochastic programming model. de Boer et al. [11] proposes a simple recourse formulation for the problem. Higle and Sen [20] models the problem as a two-stage stochastic program with nested booking limits as decision variables. They provide an extensive simulation study and state that the advantages of stochastic programming model decrease as the number of fare classes increases. Möller et al. [24] propose a stochastic programming formulation based on scenario trees, which also take cancellations into account. Cooper and Homem-de-Mello [14] develop a hybrid methodology that combines the concepts of stochastic programming and markov decision processes. Chen and Homem-de-Mello [13] propose a multi-stage stochastic programming formulation, which is hard to solve due to the lack of convexity properties. They propose an approximation scheme by solving simple recourse problems frequently.

Most available models assume that the demand is independent of the capacity controls imposed. However, customers can buy-up to a higher fare if a fare product is unavailable or can buy down if lower fare products are available. This is discussed by Belobaba [5], [4], [6] and a modified version of EMSR is proposed. Belobaba and Weatherford [7] also proposes another heuristic for formulating customer's buy up or buy down behavior. In addition, Andersson [2], [3] and Algers and Besser [1] applies logit models to estimate buy-up and recapture behavior for Scandinavian Airlines. Mahajan and van Ryzin [23] also apply logit models to analyze the product substitutions. They employ stochastic gradient methods to solve the problem. Boyd and Kallesen [12] discuss the issues of priceable and yieldable demand in a single leg setting. They argue that if customers are not properly segmented, the revenue decreases as a result of buy-downs. Cooper et al. [15] discuss the effects of buy-downs and develop a mathematical framework for optimization when buy-downs are present. Talluri and van Ryzin [29] gives a detailed analyzes of a single leg problem under general customer choice model. Zhang and Cooper [37] analyze the customer choice among parallel flights using Markov decision processes. Gallego et al.[18] generalize the idea of DLP including flexible products into the decision making scheme. They use a column generation approach to solve the problem. They also prove the asymptotic optimality of their proposed method. This idea is further analyzed by van Ryzin and Liu [31]. They propose a decomposition-type heuristic for solving the problem. van Ryzin and Vulcano [33] also incorporates the stochastic aspects of the problem and develops a stochastic gradient method for computing the virtual nesting controls.

This article focuses on three new stochastic programming formulations based on the customer's buy-up behavior. We utilize special properties of these formulations that improve the solution efficiency. The proposed formulations are compared with DLP using simulation techniques. The results indicate that using stochastic programming and modeling customer choice explicitly yield a substantial increase in expected revenue.

3 Mathematical Model Formulations

3.1 Basic Assumptions and Notation

In this section, we define our problem along with the notation. We consider an airline company operating a flight network consisting of a set of legs which is denoted by L. Each leg l has capacity c_l. The company's aim is to decide how to allocate this capacity among different itinerary/fare class combinations in order the maximize revenue. The set of itineraries using leg l is denoted by C_l and the set of all itineraries is denoted by I. Each itinerary $i \in I$ specifies the origin and destination (O&D pair) for the flight,and all legs used between origin and destination. As a result multiple itineraries may correspond to the same O&D pair. Hence, we adopt the notation I_m to denote the set of itineraries corresponding to O&D pair m and we denote the O&D pair specified by itinerary i by σ_i. The set of all O&D pairs in the network is denoted by M. The company may want to open different fare classes for different O&D pairs. We define the set $J_m = \{1, 2, \ldots, |J_m|\}$ to be the set of fare classes available for O&D pair m. We assume that 1 is the most expensive class and ticket price decreases as the index number increases.

We assume that demand is observed at O&D pair level, that is each customer arrives with a specific O&D pair and fare class in mind and chooses among available itineraries. Total stochastic demand for O&D pair m and fare class j is denoted by random variable $\tilde{d}_{mj}$. In this work, random variables are denoted with a tilde. For making the notation less cumbersome, we also assume that customers, who want to buy a ticket for O&D pair m are indifferent between the itineraries in $|J_m|$. Hence, we define the price for O&D pair m and fare class j as f_{mj}. The indifference assumption may seem restrictive, however the models in this work can be modified via minor changes if this assumption does not hold.

Even though, demand is observed at O&D pair level, the capacity allocations should be decided at itinerary level, since each itinerary is composed of different set of legs. We use x_{ij} to denote the capacity allocated for itinerary i and fare class j. These allocations are then aggregated over itineraries to find the total capacity allocated for each O&D pair. This aggregated capacity for O&D pair m and fare class j is denoted u_{mj}.

We assume that the company is risk neutral, i.e., expected revenue is maximized. We ignore the effect of cancellations and no-shows, hence we do not address the overbooking problem in this work. We also assume that capacity and demand are continuous quantities.

The stochastic programming models proposed in Section 3.2 try to incorporate buy-ups into the decision framework. We assume that if no seat is available for O&D pair m and fare class j, the customer may decide to buy a ticket from fare class $j-1$ with probability p_{mj}. The formulations in this paper imposes different assumptions on the nature of buy-ups. We discuss these assumptions and additional notation while explaining the corresponding models. The notations used in this paper is also given in Appendix for ease of reading.

3.2 Stochastic Programming Models

We start this section with a brief review of existing approaches to solve the capacity allocation problem in the revenue management context. The initial step is to analyze the problem for a single leg. The solution of the problem for a single leg when there are only two fare-classes can be characterized analytically and given in Littlewood [22]. When there are more than two classes of customers present, the system can be modelled and solved as a dynamic program. There are also efficient heuristic methods are present in the literature to approximate the problem. The capacity allocation problem for a network of resources can also be formulated as a dynamic program. However, due to the computational complexity it is almost impossible to implement the dynamic programming approach in real-time. Hence, efficient approximation schemes are developed in the literature. We can classify the existing methodologies into two groups. First approach is to decompose the network problem into a collection of single-leg problems and apply the methods used in single-leg capacity control. The second approach is to use mathematical programming models, such as deterministic linear programming(DLP) to approximate the problem. In this work, we use DLP as a benchmark for our models, as it is the most common method used in the industry. Hence, we give an overview of DLP and refer the reader to the book by Talluri and van Ryzin [30] for a review of other methodologies.

The deterministic linear programming (DLP) approach formulates the capacity allocation problem using the expected value of the demand vector. The classical formulation of the DLP assumes that each customer demands a ticket for a specific itinerary and fare class pair and leaves the system if no capacity is available for that itinerary and fare class pair. In this work, we assume that each customer demands a ticket for a specific O&D pair and fare class, but she is indifferent between different itineraries corresponding to that specific O&D pair. This indifference assumption is the same as assuming company has flexible products for each O&D pair and leads the same formulation as in Gallego et al. [18]. Finally, the decision variables of DLP are partitioned booking limit controls.

In the following model, the variables x_{ij} denotes the partitioned booking limit assigned to itinerary i and fare class j. Constraint (1b) assures that no more than the available capacity can be sold from each leg l. Constraint (1c) aggregates the booking limits allocated for the itineraries to find the total booking limit for the O&D pairs. The total booking limit for O&D pair m and fare class j is denoted u_{mj}. Since demand is assumed to be known, this is also equal to the number of tickets sold for the corresponding O&D pair and fare class and therefore bounded by the expected value of the demand.

$$\max_{x,u} \quad \sum_{m\in M}\sum_{j\in J_m} f_{mj}u_{mj} \tag{1a}$$

$$\text{s.t.:} \quad \sum_{i\in C_l}\sum_{j\in J_{\sigma_i}} x_{ij} \leq c_l, \qquad \forall l \in L \tag{1b}$$

$$\sum_{i\in I_m} x_{ij} = u_{mj}, \qquad \forall m \in M, \forall j \in J_m \tag{1c}$$

$$0 \leq u_{mj} \leq E(\tilde{d}_{mj}), \qquad \forall m \in M, \forall j \in J_m \tag{1d}$$

$$x_{ij} \geq 0, \qquad \forall i \in I, \forall j \in J_{\sigma_i} \tag{1e}$$

For real life applications, the booking limits obtained from the DLP can be used. However, a more common practice is to use DLP to generate the bid prices. One method is to generate bid prices using the dual variables corresponding to the constraints (1b). There also exists other techniques to generate bid prices. To employ bid price controls effectively, one should re-optimize DLP frequently while updating the parameters.

In this section, we give three stochastic programming formulations that addresses two of the shortcomings of DLP. Firstly, DLP uses only the first moment information about the demand. We can hope to improve our results by making better use of the distributional information. Secondly, DLP assumes that customers demand a specific fare class and leave the system if that fare class is not available. However, in reality customers may tend to purchase from a more expensive fare class, if there is no seat available for their original choice.

In the stochastic formulations, it is assumed that the control is static, i.e., the booking limits are set before the demand is realized and the company operates under those booking limits without updating them. This structure can be easily modelled via a two-stage approach which is common in stochastic programming. In the first stage problem, the goal is to determine the booking limits, whereas the second stage provides us with an approximation of revenue generated given a specific demand and booking limit vector. The first stage problem is given below:

$$
\begin{aligned}
\max_{x,u} \quad & E(h(u,\tilde{d})) && && \text{(2a)}\\
\text{s.t.:} \quad & \sum_{i \in C_l} \sum_{j \in J_{\sigma_i}} x_{ij} \le c_l, && \forall l \in L && \text{(2b)}\\
& \sum_{i \in I_m} x_{ij} = u_{mj}, && \forall m \in M, \forall j \in J_m && \text{(2c)}\\
& u_{mj} \ge 0, && \forall m \in M, \forall j \in J_m && \text{(2d)}\\
& x_{ij} \ge 0, && \forall i \in I, \forall j \in J_{\sigma_i} && \text{(2e)}
\end{aligned}
$$

This first stage formulation has the same structure with DLP with two major differences. Firstly, instead of maximizing a linear objective, this problem maximizes the expected value of a generic function $h(u,\tilde{d})$, which is given by the second stage formulation. Secondly, demand is not explicitly modelled in the first stage constraints, since it is taken into account in the objective. This generic definition of $h(u,\tilde{d})$ provides us some freedom in modelling the demand generation scheme. The formulations given below exploit this freedom to model different assumptions.

The first formulation given below assumes that company is operating under partitioned booking limit controls. The customers arriving for O&D pair m and fare class j may choose to buy-up with probability p_{mj}. If a customer decides to buy-up, then she behaves like the upper fare class customer, i.e., if no seat is available from upper fare class, she may continue buying up with probability corresponding to her new class. For the formulation given below,

v^1_{mj} is the amount of tickets sold for O&D pair m and fare class j and v^2_{mj} is the unmet demand. To model our assumption on customer behavior, the demand resulting from buy-ups is added to the original demand of class j on the righthand side of (3b), hence the model does not distinguish people who originally demand class j from the people who bought up. Constraint (3d) assures that we do not sell more than the booking limit. Throughout this work, (2) coupled with (3) is referred as SLP1.

$$
\begin{aligned}
h(u,\tilde{d}) \quad &= \quad \max_{v^1,v^2} \sum_{m\in M}\sum_{j\in J_m} f_{mj}v^1_{mj} && && \text{(3a)}\\
\text{s.t:} \quad & v^1_{mj} + v^2_{mj} = \tilde{d}_{mj} + p_{mj+1}v^2_{mj+1} && \forall m \in M, j < |J_m| && \text{(3b)}\\
& v^1_{m|J_m|} + v^2_{m|J_m|} = \tilde{d}_{m|J_m|} && \forall m \in M && \text{(3c)}\\
& v^1_{mj} \le u_{mj} && \forall m \in M, \forall j \in J_m && \text{(3d)}\\
& v^1_{mj}, v^2_{mj} \ge 0 && \forall m \in M, \forall j \in J_m && \text{(3e)}
\end{aligned}
$$

Different from SLP1, the second formulation assumes that customers only buy-up once and leave the system if no ticket is available from the upper fare class. To assure this, the demand resulting from buy-ups is not added to original demand, but taken into account in capacity constraint (4c). We refer (4) as SLP2 in the remainder of this work.

$$
\begin{aligned}
h(u,\tilde{d}) \quad &= \quad \max_{v^1,v^2} \sum_{m\in M}\sum_{j\in J_m} f_{mj}(v^1_{mj} + p_{mj+1}v^2_{mj+1} - s_{mj}) && && \text{(4a)}\\
\text{s.t:} \quad & v^1_{mj} + v^2_{mj} = \tilde{d}_{mj} && \forall m \in M, j \in J_m && \text{(4b)}\\
& v^1_{mj} + p_{mj+1}v^2_{mj+1} - s_{mj} \le u_{mj} && \forall m \in M, j < |J_m| && \text{(4c)}\\
& v^1_{m|J_m|} \le u_{m|J_m|} && \forall m \in M, && \text{(4d)}\\
& v^1_{mj}, v^2_{mj} \ge 0 && \forall m \in M, \forall j \in J_m && \text{(4e)}
\end{aligned}
$$

As in SLP1, the third formulation assumes that customers can buy-up more than once. However, instead of using partitioned booking limits, this formulation uses theft nesting controls. For this purpose $\sum_{i=j}^{|J_m|} u_{mi}$ is treated as booking limit of class j. Since the controls are nested, we have to estimate the actual capacity used by class j. In the current formulation, this reduces to finding out what is the percentage of u_{mi} sold to class j customers for a given $\tilde{d}$. If we assume that arrivals follow a homogeneous poisson process, we can utilize the order statistics property of arrivals to estimate this percentage. Order statistics property tells us that if the demand for each class is known, the arrivals can be generated using a uniform distribution. We refer the reader to [21] for a detailed explanation of order statistics property of poisson processes. Using this result, we estimate the percentage u_{mi} sold to class j customers as $\tilde{r}_{mij} = \tilde{d}_{mj} / \sum_{k \le i} \tilde{d}_{mk}$. Since $\tilde{r}_{mij}$ is a function of $\tilde{d}$, it is also a random variable. These assumptions yields the following formulation, which is referred as SLP3 in the remainder of this work.

$$h(u, \tilde{d}) = \max_{v^1, v^2} \sum_{m \in M} \sum_{j \in J_m} f_{mj} v^1_{mj} \tag{5a}$$

$$\text{s.t:} \quad v^1_{mj} + v^2_{mj} = \tilde{d}_{mj} + p_{mj+1} v^2_{mj+1} \quad \forall m \in M, j < |J_m| \tag{5b}$$

$$v^1_{m|J_m|} + v^2_{m|J_m|} = \tilde{d}_{m|J_m|} \quad \forall m \in M \tag{5c}$$

$$v^1_{mj} \le \sum_{i \in J_m, i \ge j} \tilde{r}_{mij} u_{mi} \quad \forall m \in M, \forall j \in J_m \tag{5d}$$

$$v^1_{mj}, v^2_{mj} \ge 0 \quad \forall m \in M, \forall j \in J_m \tag{5e}$$

4 Solution Methodology

The two-stage nature of the stochastic programs makes our lives easier in terms of modelling the uncertainty and customer choice. However, it also brings out a question about tractability of the formulations. In this section, we discuss how to apply stochastic programming algorithms to solve our problem. We also point out some of the structural properties of our formulations that may improve efficiency of these algorithms.

By setting the buy-up probabilities to 0 and replacing random demand with the expected value, SLP1 and SLP2 reduces to DLP. In the same manner, if we replace the random demand in SLP1 with its expected value, we get the following deterministic linear program:

$$
\begin{aligned}
\max_{x,u} \quad & \sum_{m\in M}\sum_{j\in J_m} f_{mj}u_{mj} && && \text{(6a)}\\
\text{s.t.:} \quad & \sum_{i\in C_l}\sum_{j\in J_{\sigma_i}} x_{ij} \le c_l, && \forall l \in L && \text{(6b)}\\
& \sum_{i\in I_m} x_{ij} = u_{mj}, && \forall m\in M, \forall j \in J_m && \text{(6c)}\\
& v^1_{mj} + v^2_{mj} = E(\tilde{d}_{mj}) + p_{mj+1}v^2_{mj+1} && \forall m \in M, j < |J_m| && \text{(6d)}\\
& v^1_{m|J_m|} + v^2_{m|J_m|} = E(\tilde{d}_{m|J_m|}) && \forall m \in M && \text{(6e)}\\
& v^1_{mj} \le u_{mj} && \forall m\in M, \forall j \in J_m && \text{(6f)}\\
& u_{mj}, v^1_{mj}, v^2_{mj} \ge 0, && \forall m\in M, \forall j \in J_m && \text{(6g)}\\
& x_{ij} \ge 0, && \forall i \in I, \forall j \in J_{\sigma_i} && \text{(6h)}
\end{aligned}
$$

Formulation (6) is referred as DLPB in the remainder of this work.

Theorem 4.1. *Let z^*_{SLP1} and z^*_{DLPB} be the optimal values of SLP1 and DLPB respectively. Then $z^*_{SLP1} \le z^*_{DLPB}$*

Proof. To show that $z_{SLP1} \le z_{DLPB}$, we show that $h(u, \tilde{d})$ given in (3) is concave in $\tilde{d}$. For this purpose, we take the dual of (3). To simplify the notation the feasible set of the dual is denoted by Π. Letting η and π be the dual variables corresponding to the constraints (3d) and (3c, 3b) respectively, we write

$$h(u, \tilde{d}) = \min_{(\eta,\pi)\in\Pi} \sum_{m\in M}\sum_{j\in J_m} \eta_{mj}u_{mj} + \pi_{mj}\tilde{d}_{mj} \tag{7}$$

Hence for $\lambda \in (0,1)$,

$$\begin{aligned} h(u, \lambda\tilde{d}^1 + (1-\lambda)\tilde{d}^2) &= \min_{(\eta,\pi)\in\Pi} \sum_{m\in M} \sum_{j\in J_m} \eta_{mj}u_{mj} + \pi_{mj}(\lambda\tilde{d}^1_{mj} + (1-\lambda)\tilde{d}^2_{mj}) \\ &= \min_{(\eta^1,\pi^1)=(\eta^2,\pi^2)\in\Pi} \sum_{m\in M} \sum_{j\in J_m} \lambda(\eta^1_{mj}u_{mj} + \pi^1_{mj}\tilde{d}^1_{mj}) \\ &\quad +(1-\lambda)(\eta^2_{mj}u_{mj} + \pi^2_{mj}\tilde{d}^2_{mj}) \\ &\geq \lambda \min_{(\eta,\pi)\in\Pi} \sum_{m\in M} \sum_{j\in J_m} \eta_{mj}u_{mj} + \pi_{mj}\tilde{d}^1_{mj} \\ &\quad +(1-\lambda) \min_{(\eta,\pi)\in\Pi} \sum_{m\in M} \sum_{j\in J_m} \eta_{mj}u_{mj} + \pi_{mj}\tilde{d}^2_{mj} \\ &= \lambda h(u,\tilde{d}^1) + (1-\lambda)h(u,\tilde{d}^2), \end{aligned}$$

which proves the concavity of $h(u,\tilde{d})$. Let (x^*, u^*) be an optimal solution of SLP1,

$$\begin{aligned} z^*_{SLP1} &= E_{\tilde{d}} h(u^*, \tilde{d}) \\ &\leq h(u^*, E_{\tilde{d}}(\tilde{d})) \\ &\leq z_{DLPB}. \end{aligned}$$

The first inequality above follows from Jensen's inequality and the second inequality from feasibility of (x^*, u^*) for the deterministic problem. □

By using the concavity of the second stage function $h(\cdot)$, it is also possible to come up with lower bounds on the optimal value z^*_{SLP1}. However, the analysis for the lower bound is more involved and we refer the reader to [10] for the details. Similar results can also be obtained for SLP2 and SLP3.

If the random demand vector $\tilde{d}$ has finite support, i.e., there are only finitely many possible scenarios, we may write the problem as a large scale equivalent LP, where we combine first and second stage formulations. However, such an approach is not very attractive, since the solution time of the LP scales cubically in terms of the number of scenarios. To overcome this problem, L-shaped method which is developed by van Slyke and Wets [34] can be used. L-shaped method is a cutting plane algorithm and exploits the concavity and piecewise linearity of the function $h(u,\tilde{d})$ with respect to u. In L-shaped method, the function $h(u,\tilde{d})$ is approximated by using gradients based on the optimal dual solutions of second stage problem at each iteration. For more information on L-shaped method, we refer the reader to [10].

When demand has an infinite(possibly uncountable) support, then the methods discussed in the above paragraph cannot be used directly. An approach to tackle this issue is to use Monte Carlo sampling idea and try to approximate the distribution by drawing N sample demand vectors from the joint distribution. Then L-shaped method can be applied to the N-sample problem to minimize the sample mean. This approach is completely probabilistic and depends very much on the random sample. However, it can be proven that this method yields asymptotically optimal controls as we increase the sample size N and the rate at which the error converges to zero only depends on N, i.e., independent of the dimension of the demand vector. Shapiro [26] gives an excellent review of the Monte Carlo methods used in stochastic optimization.

To obtain a better solution, S replications of N-sample problems are generated and the optimal solutions of these S problems are averaged to get the optimal controls. When buy-up probability is 0 and $N = 1$, this method reduces to the randomized linear programming algorithm(RLP), which is commonly used in revenue management literature. Although, it seems as a simple change in parameters, choosing $N > 1$ shows a significant change in the mind setting of the decision maker. If $N = 1$, the decision maker makes a myopic decision to optimize the system for that specific demand realization. This is called *wait-and-see solution* in stochastic programming literature. Whereas, if $N > 1$, the decision maker takes different realization into account and makes the decision to optimize the expected profit, using more information on demand. Birge and Louveaux [10] provides an excellent discussion on the superiority of stochastic solution to wait-and-see solution.

In practice, the controls are optimized frequently as demand reveals. This allows the decision makers to take reactive actions. Let $t_1 < t_2 < \cdots < T$ be the time points when the model is re-optimized and T is the time of take-off. We assume that fares are static and the revenue is managed by controlling the availability of fare classes. Hence, when re-optimizing, the distribution of demand and capacity changes and the other parameters stays the same.

The change in capacity is reflected to the second stage decision by the change in allocation vector. A careful examination of the second stage problem shows that the parameters that change by time only shows up on the righthand side vector of the LP. Hence, the dual feasible region does not change. Remembering that, the cuts in the L-shaped method are generated by using the extreme points of the dual feasible region for the second stage problem, we have the following observation:

Observation 4.2. *Assuming that the fares are static, the dual feasible region for the second stage problem is the same for all re-optimization points. Hence, the cuts generated at* t_i *can be used at time point* $t_j, j > i$ *to decrease the computation time.*

Using Observation 4.2 reduces the number of major iterations performed by the L-shaped method, hence it is expected to accelerate the computation process.

5 Simulation Methodology and Numerical Results

Stochastic programming models introduced in this work, address some of the shortcomings of DLP. Nevertheless, these models are stationary models and do not capture the dynamic nature of the allocation process. Therefore, the proposed methods should be tested on a real time simulation.

To imitate the real time ticket selling process we designed a simulation experiment and assess the results on a toy problem. The toy problem used in this work is composed of 4 cities and 5 legs. The layout of the network is given in Figure 1. The numbers in Figure 1 shows the seat capacity of the corresponding leg. This network serves the customers from 6 O&D pairs and there are a total of 10 itineraries available. The arriving customers choose between three fare classes, namely Business, Leisure-1 and Leisure-2. The arrivals are assumed to follow a homogeneous poisson process for each class. The data for expected number of arrivals and price corresponding to each O&D pair and fare class is given in Table 1.

To solve the stochastic programs, L-shaped method was utilized combined with Monte Carlo sampling. For each program 50 sample demand points were generated. To obtain the

O&D	Fare Class	Fare	Arrival Rate
Austin-Dallas	Business	\$500	10
	Leisure-1	\$120	30
	Leisure-2	\$50	50
Austin-Chicago	Business	\$800	15
	Leisure-1	\$400	35
	Leisure-2	\$200	100
Austin-New York	Business	\$1200	8
	Leisure-1	\$450	40
	Leisure-2	\$250	70
Dallas-Chicago	Business	\$900	15
	Leisure-1	\$420	40
	Leisure-2	\$320	110
Dallas-New York	Business	\$1000	15
	Leisure-1	\$375	50
	Leisure-2	\$230	90
Chicago-New York	Business	\$650	10
	Leisure-1	\$275	60
	Leisure-2	\$150	75

Table 1: Problem Data

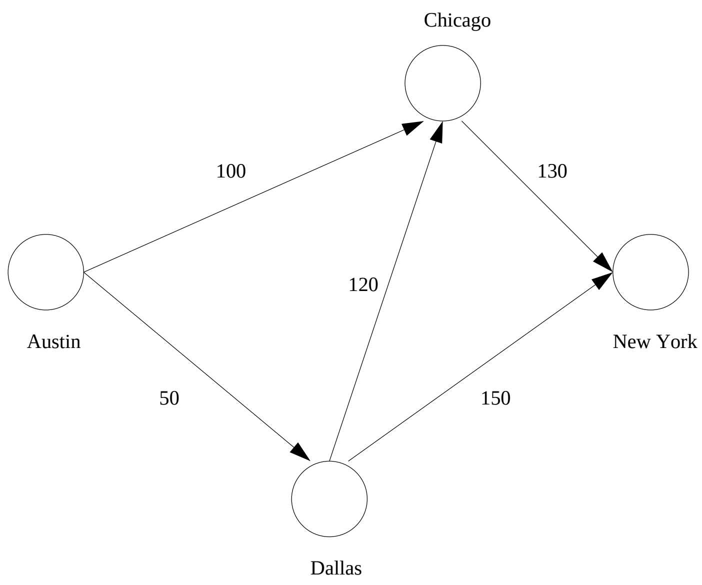


Figure 1: Network with 5 legs and 6 O&D pairs

controls, 100 replications of 50-sample problems were solved. The booking limit controls were employed to generate revenue. A heuristic method was used to convert non-integer solutions to integer booking limits. To avoid experimental bias, the demand points generated for Monte Carlo method and the demand in real time simulation are generated using independent random number streams.

We first tested our formulations with a single optimization scheme. The booking limits were determined for each formulation before the simulation run and the tickets are sold without re-optimization as demand reveals. For SLP1, SLP2, DLPB and DLP nested booking limit controls are used. However, due to the explicit modelling of theft nesting, SLP3 was tested using theft nesting controls. The customers are allowed to buy-up more than once, if there is no capacity available from the upper fare class. The formulations are tested for buy-up probabilities ranging between 0 and 0.3. Each simulation is replicated 100 times. Figure 2 shows the expected percentage increase in the revenue with respect to DLP. When buy-up probability is 0, SLP3 performs the best, resulting around 1.5% increase in revenue. As buy-up probability increases SLP1 performs better, yielding a 10% revenue increase when buy-up

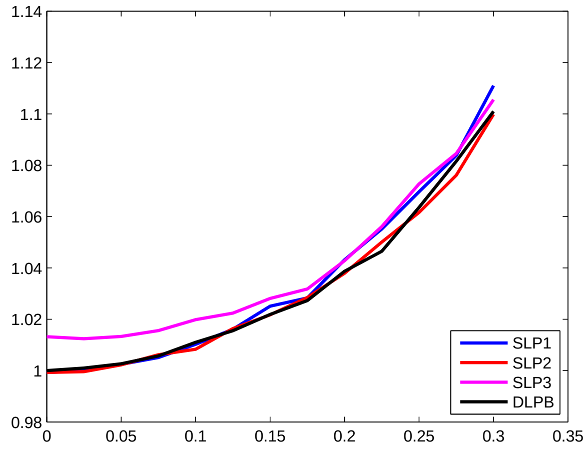
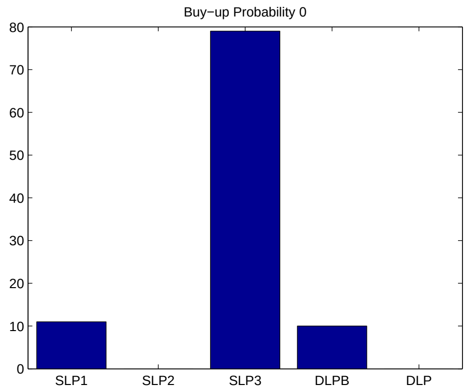


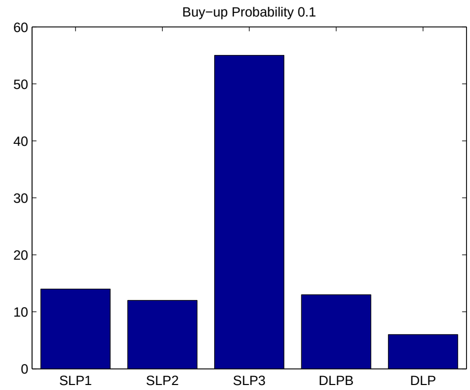
Figure 2: Expected percentage w.r.t. DLP under behavior assumption of SLP1: Single optimization case

probability is 0.3. Also this increase in revenue gain shows how important it is to model buy-up behavior in network revenue management. The bar graph presented in Figure 3 shows the number of replications each method had revenue advantage over others. When buy-up probability is 0, SLP3 had a revenue advantage in 80 of 100 replications. However, when buy-up probability is 0.3, SLP1 beat other methods in 40% of all replications. These results are consistent with the results demonstrated in Figure 2.

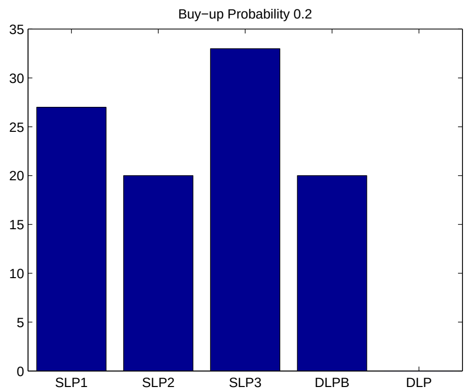
The results presented above show that the stochastic programming approach performs better than the deterministic linear program when the optimization is done at the beginning of the booking period. However, in practice the booking limits are updated frequently as the demand reveals. This frequent optimization approach increases the performance of DLP significantly(see e.g. [35]). To assess the performance of our formulations in this setting, we assumed that the optimization process starts 40 days prior to departure and booking limits are updated every night. Figure 4 shows the expected percentage increase in revenue and Figure 5 shows the number of replications which each method had revenue advantage over others. When buy-up probability is 0, DLPB outperforms other methods 55% of the replications. However, by examining Figure 4, we see that the expected revenue is almost the



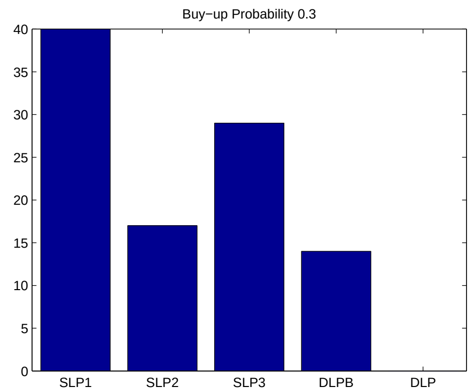
(a) $p = 0$



(b) $p = 0.1$



(c) $p = 0.2$



(d) $p = 0.3$

Figure 3: Number of replications dominated by each model under behavior assumption of SLP1: Single optimization case

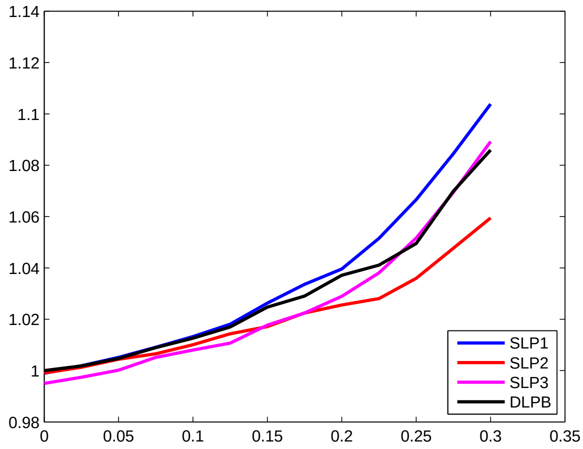
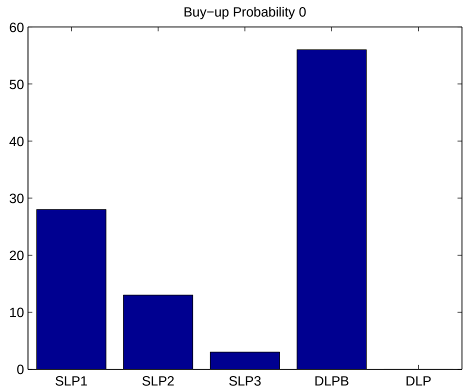


Figure 4: Expected percentage w.r.t. DLP under behavior assumption of SLP1: Frequent optimization case

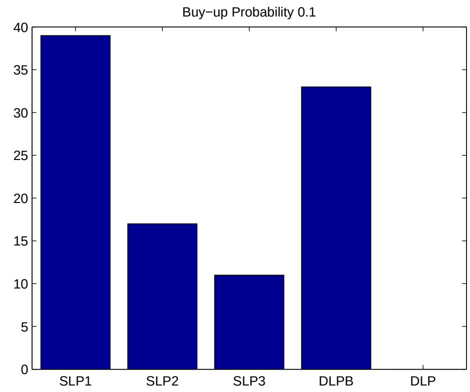
same. As buy-up probability increases the value of the stochastic solution increases. When buy-up probability is 0.3, we see that SLP1 outperforms other methods in more than 60% percent of all replications and yields a revenue gain more than 10% over DLP.

When modelling SLP2 it is assumed that customers buy-up only once and leave the system if no ticket is available from that upper class. To test how proposed formulations perform under this assumption, the simulation model described above is modified. It is assumed that the optimization process starts 20 days prior to take-off and the booking limits are updated every night. Figure 6 shows the expected percentage increase in revenue and Figure 7 shows the number of replications which each method had revenue advantage over others. Figure 6 shows that all the methods has almost the same performance, however as buy-up probability increases the value of stochastic solution increases. Interestingly, for low buy-up probabilities, SLP2 performs worse than SLP1 and DLPB, but as the buy-up probability increases, modelling the actual behavior pattern becomes more and more important and when buy-up probability is 0.3, SLP2 beats other formulations in 40% of all the replications.

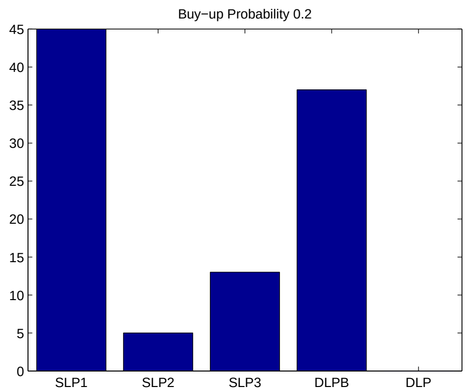
For the aforementioned simulation models, it was assumed that the customers arrive



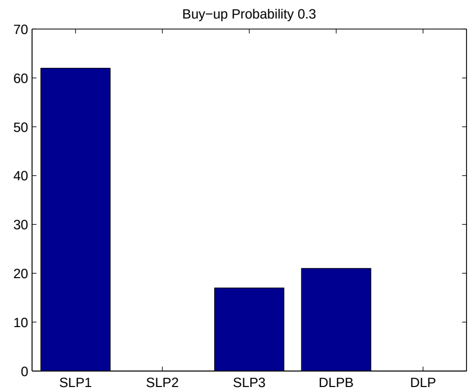
(a) $p = 0$



(b) $p = 0.1$



(c) $p = 0.2$



(d) $p = 0.3$

Figure 5: Number of replications dominated by each model under behavior assumption of SLP1: Frequent optimization case

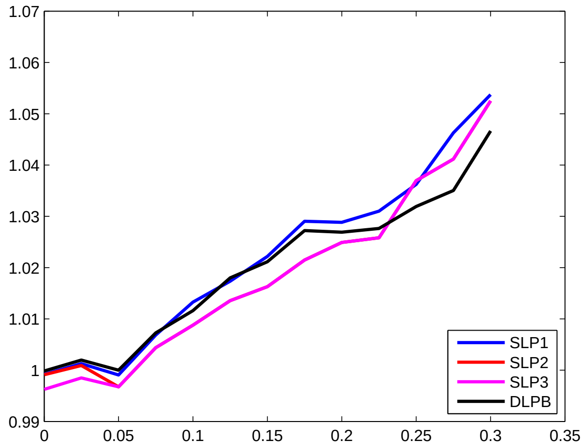
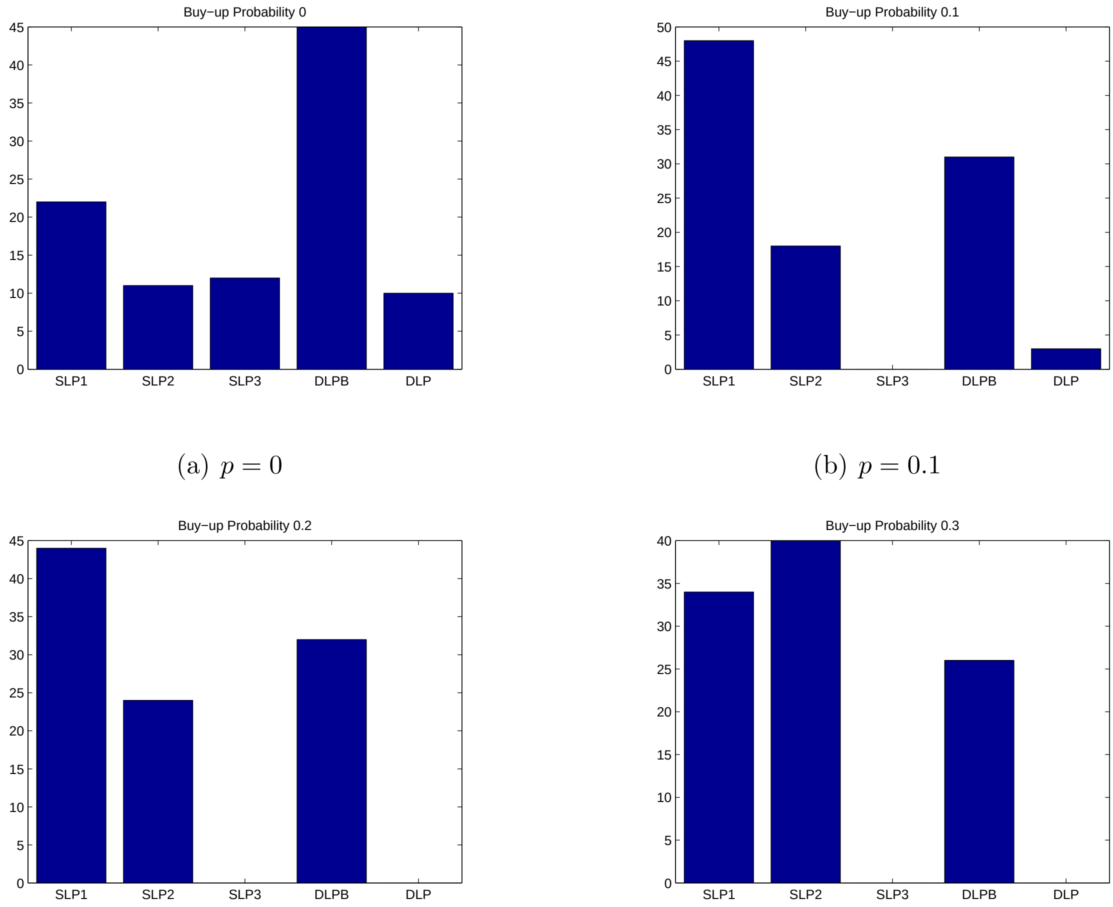


Figure 6: Expected percentage w.r.t. DLP under behavior assumption of SLP2: Frequent optimization case

demanding a specific fare class and they tend to buy-up if there is no seat available from that class. However, in reality a business customer may tend to buy a leisure ticket, if there is any ticket available from leisure class. This issue is called buy-down behavior in revenue management literature. Although this behavior is not explicitly modelled, we tested our formulations in the presence of buy-down behavior. In this simulation model, customers arrive to the system with a specific fare class in mind but they buy a lower fare if available. If there is no seat available for any of the lower fares and the fare they demand originally, then they make a buy-up decision. The customers can buy-up more than once, if there is no seat available for the upper class. It is assumed that the optimization process starts 20 days prior to take-off and the booking limits are updated every night. Figure 8 shows the expected percentage increase in revenue and Figure 9 shows the number of replications which each method had revenue advantage over others. As in the previous experiments, the value of stochastic solution increases as buy-up probability increases. However, when the customers are assumed to buy-down, the revenue gain with stochastic solution is more important compared to the previous experiments.



(a) $p = 0$

(b) $p = 0.1$

(c) $p = 0.2$

(d) $p = 0.3$

Figure 7: Number of replications dominated by each model under behavior assumption of SLP2: Frequent optimization case

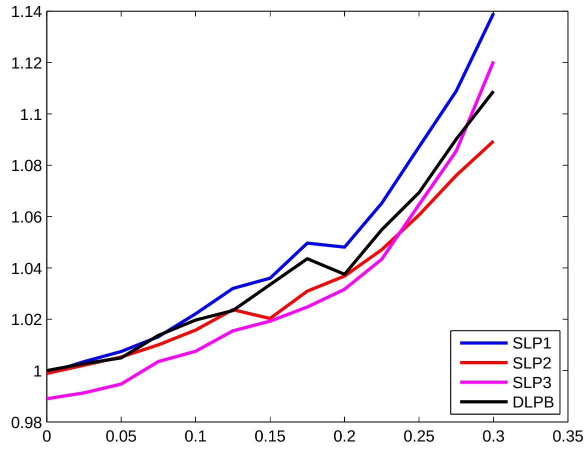
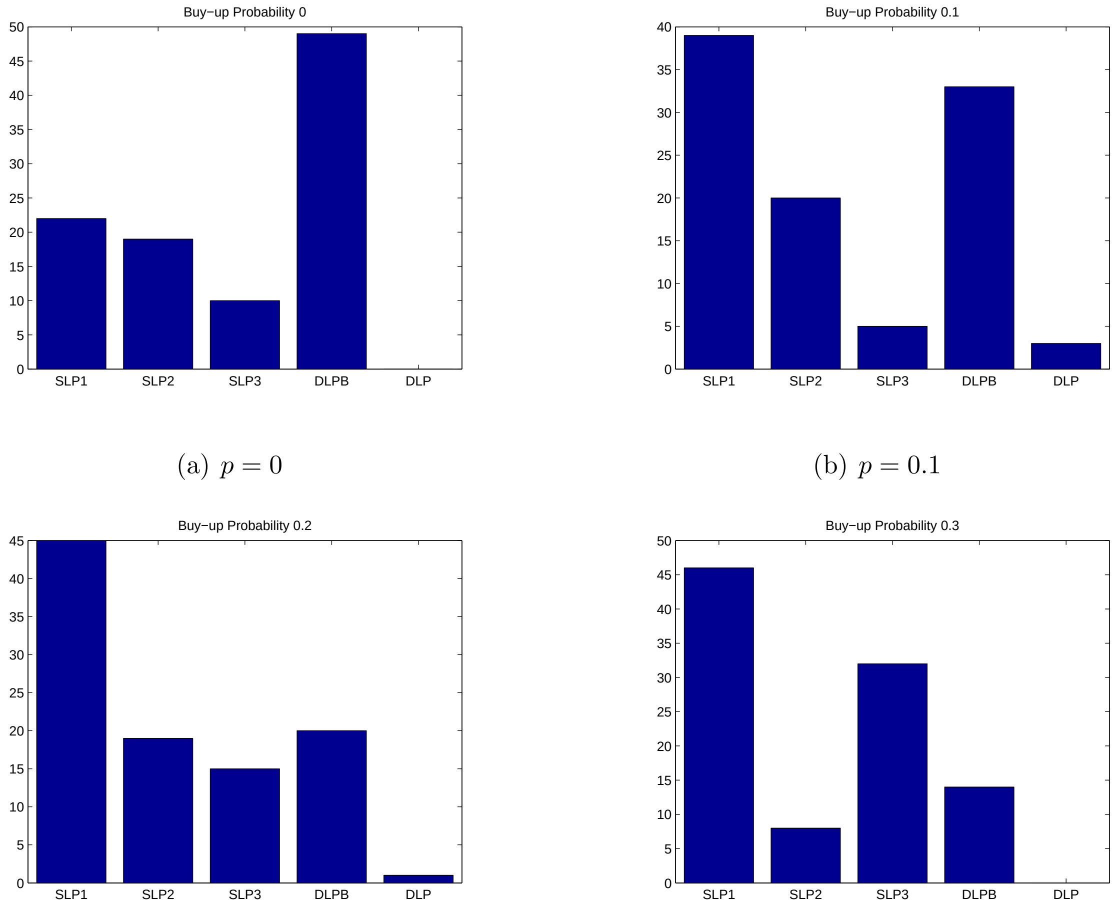


Figure 8: Expected percentage w.r.t. DLP in the presence of buy-downs

6 Conclusion and Future Work

In this paper, we addressed two important issues arising in modelling of network capacity allocation problems. We developed three stochastic modelling formulations for the network capacity control problem, taking buy-up decisions into account. Our results indicate that capturing the buy-up behavior of the customers in the model yields considerable increase in the revenue, even when stochasticity in the demand is not captured. Our results also show that the importance of stochastic modelling increases when customers are more willing to buy-up. The formulations are also tested when the customers buy the lowest fare available, (i.e., in the presence of buy-downs). We observed that the stochastic programming models yield a better revenue than the deterministic linear programming approach.

At this point, there are three clear cut ways to improve the model. So far we have assumed that customers are arriving individually. One way to extend this problem is modelling the problem, when arrivals can occur in batches. Another important problem in revenue management is to model the overbooking decisions. Due to our ability in modelling the probabilistic structure, we also think that overbooking decisions can also be modelled as a part of the network revenue management problem.



(a) $p = 0$

(b) $p = 0.1$

(c) $p = 0.2$

(d) $p = 0.3$

Figure 9: Number of replications dominated by each model in the presence of buy-downs

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